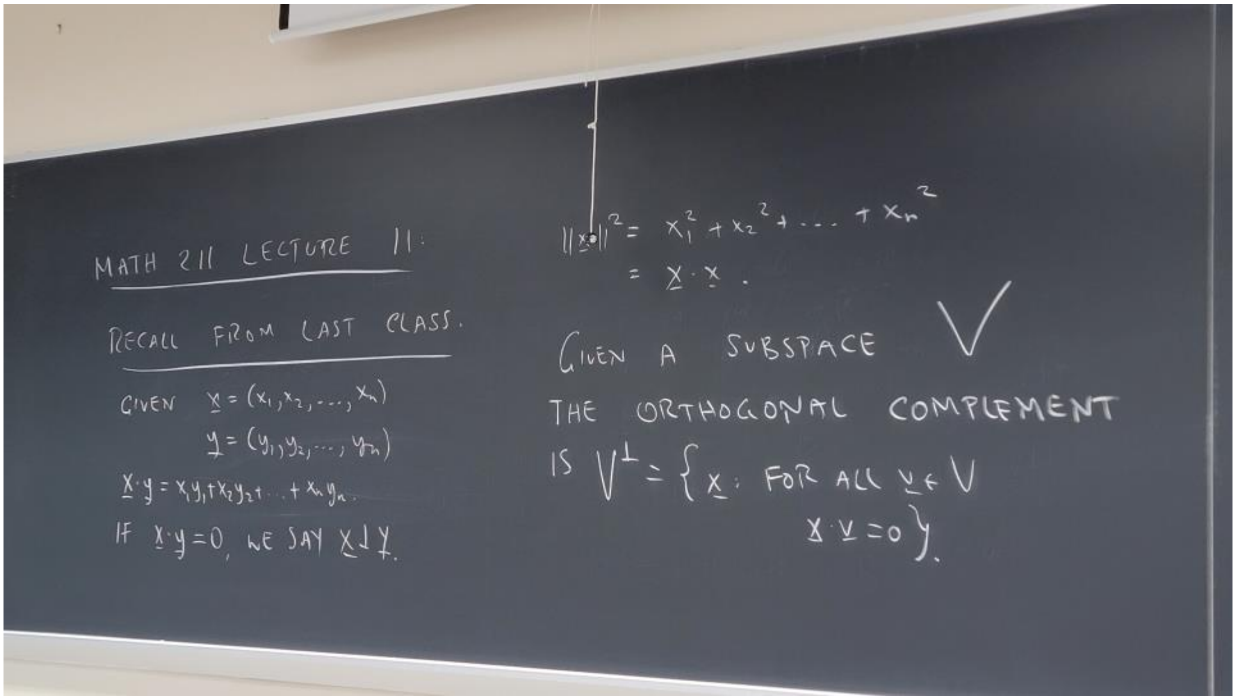
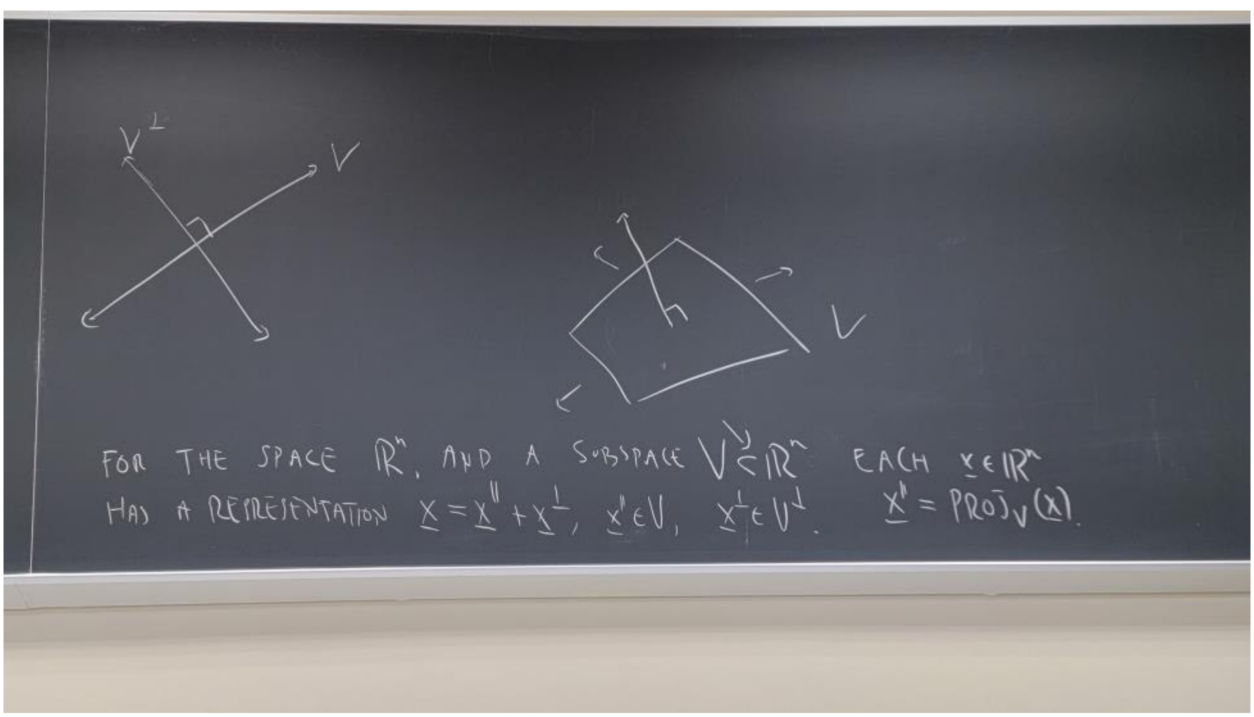
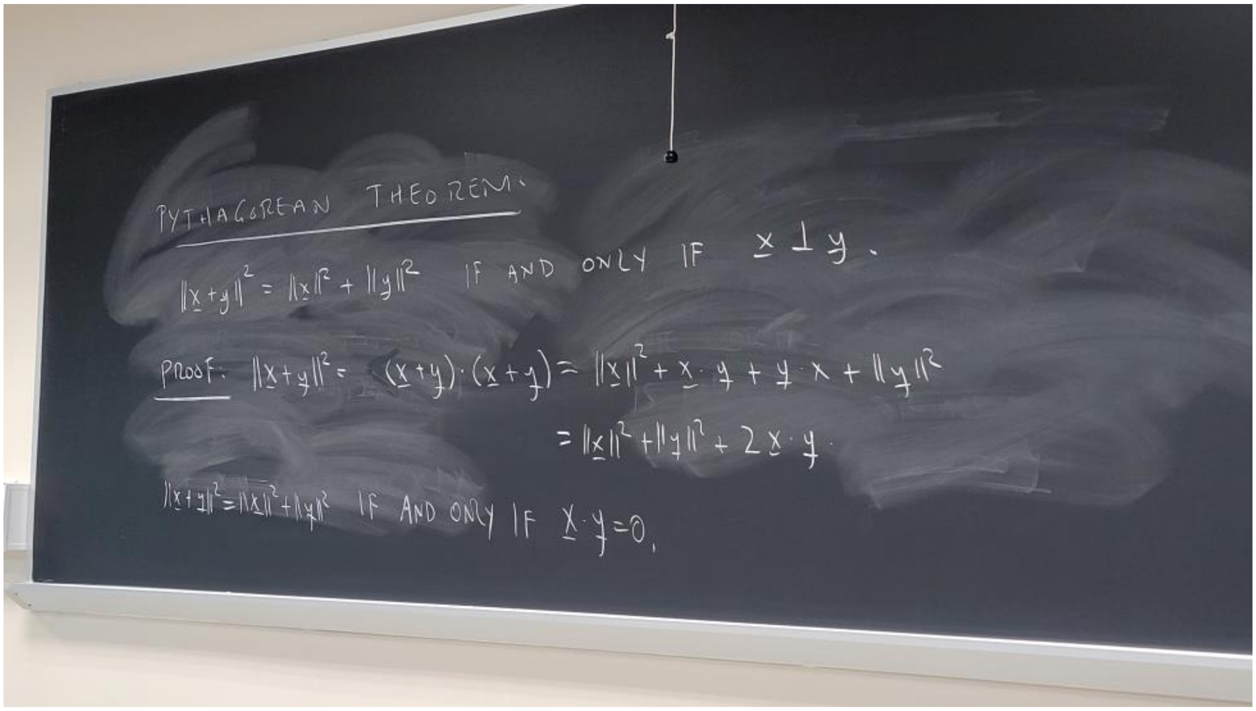
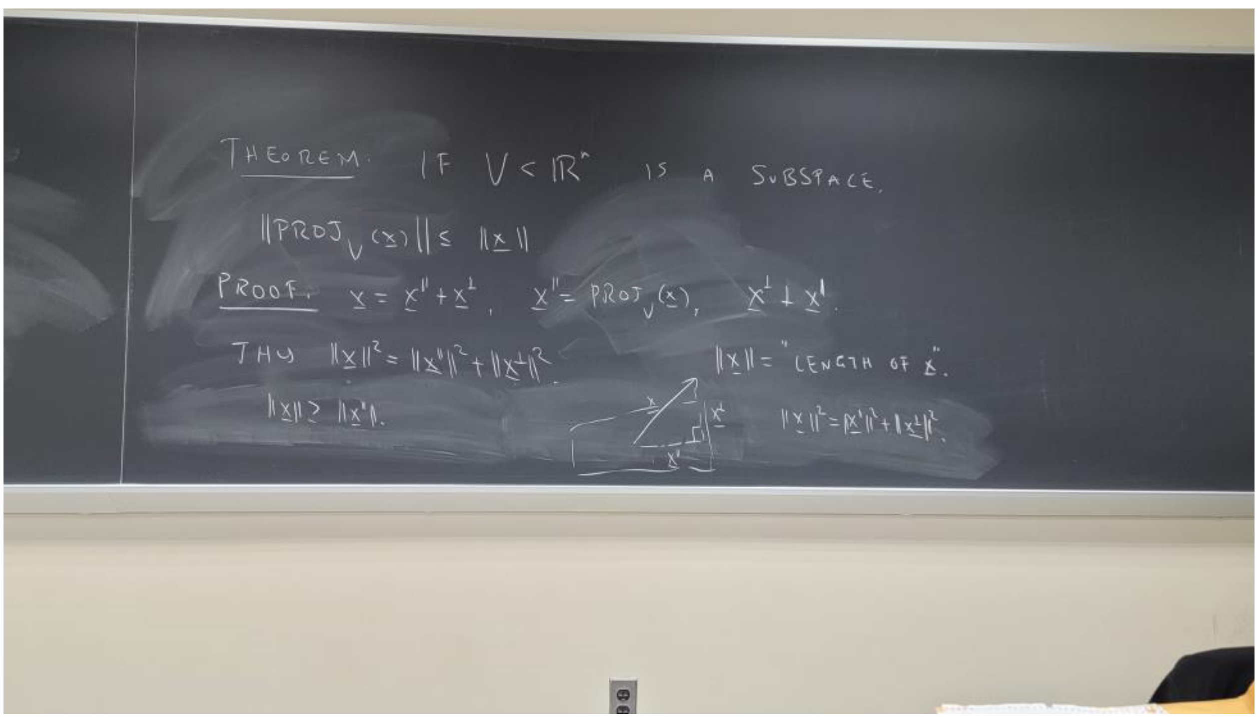
3/2/23

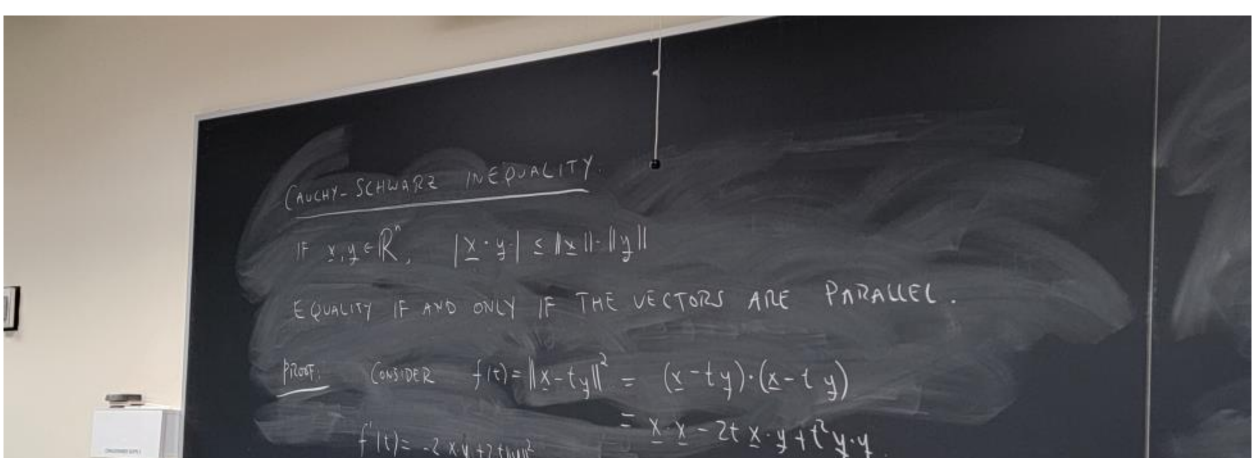
Thursday, March 9, 2023 3:32 PM

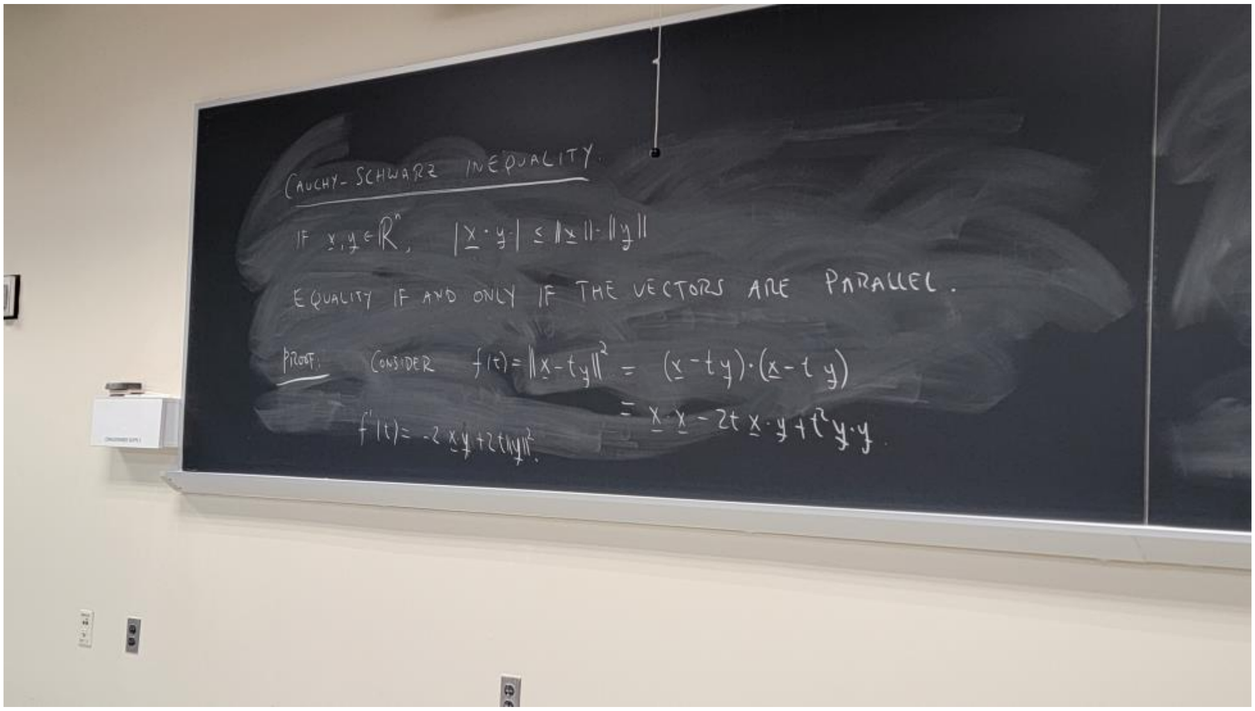


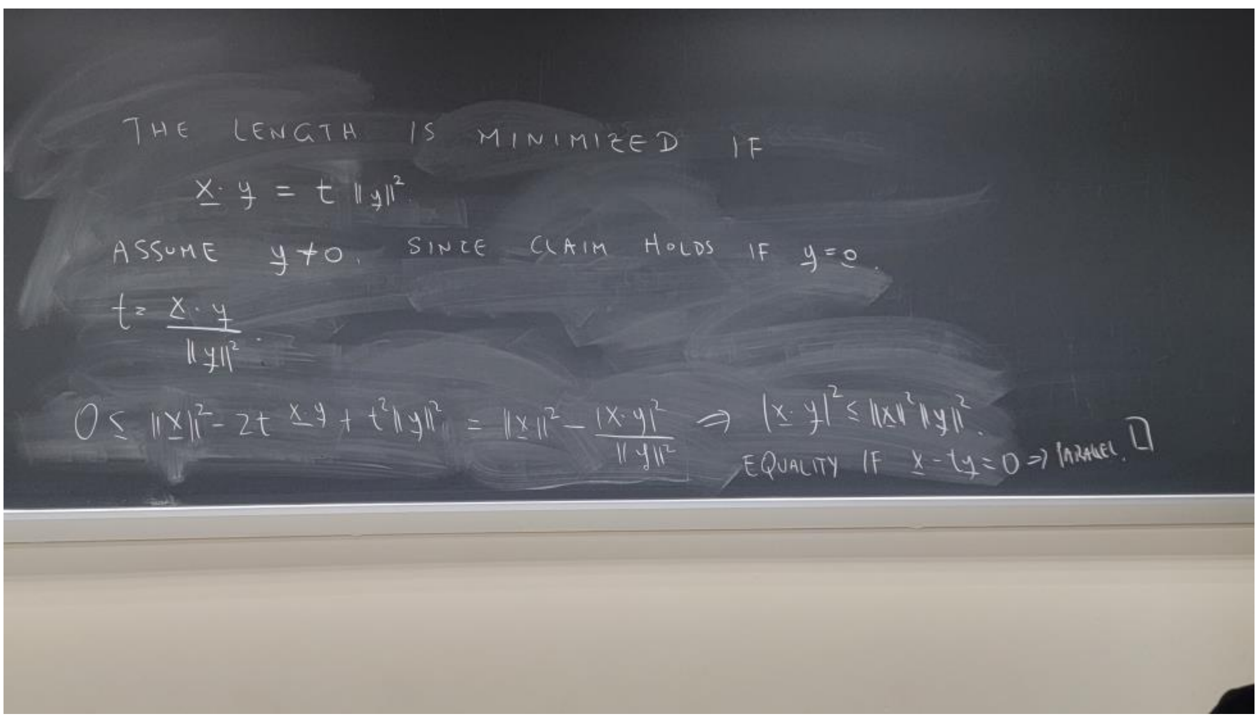


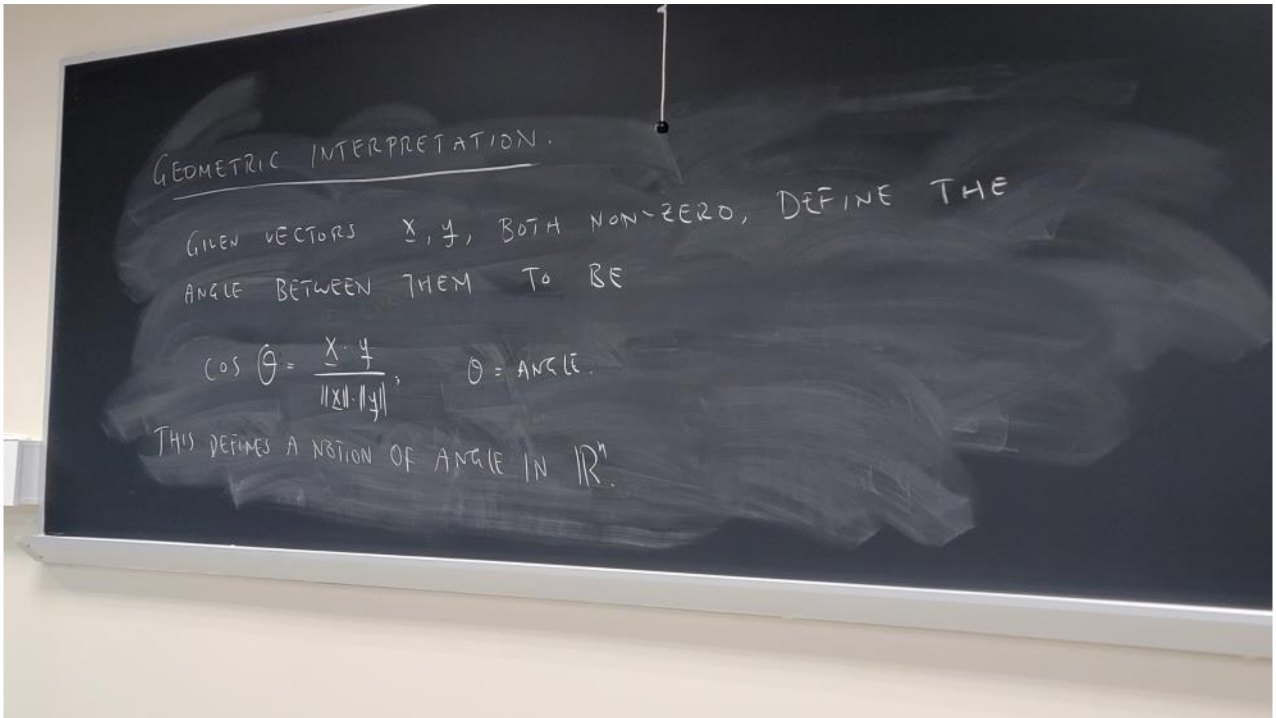

PYTHAGOREAN THEOREM:
$\|\underline{x}+\underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2$ IF AND ONLY IF $\underline{x} \perp \underline{y}$.
PROOF: $\|\underline{x}+\underline{y}\|^2 = (\underline{x}+\underline{y})\cdot(\underline{x}+\underline{y}) = \|\underline{x}\|^2 + \underline{x}\cdot\underline{y} + \underline{y}\cdot\underline{x} + \|\underline{y}\|^2$
$= \|\underline{x}\|^2 + \|\underline{y}\|^2 + 2\,\underline{x}\cdot\underline{y}$.
$\|\underline{x}+\underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2$ IF AND ONLY IF $\underline{x}\cdot\underline{y} = 0$.

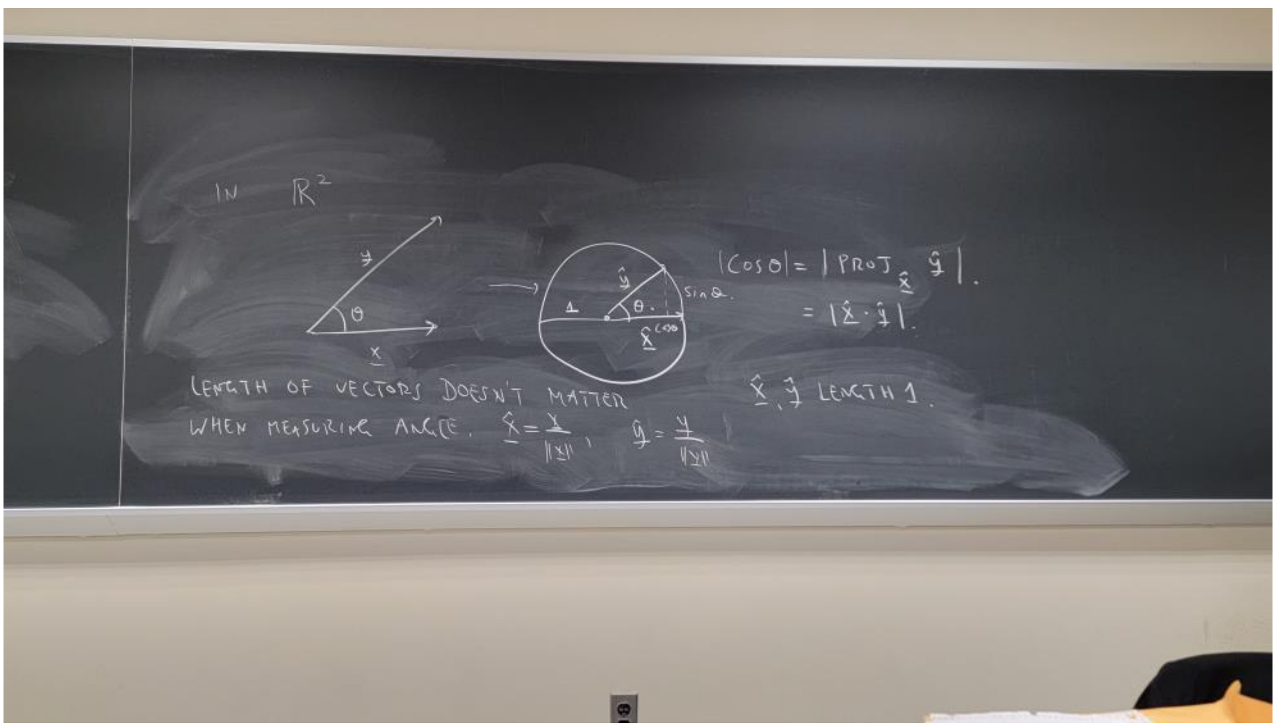

THEOREM: IF $V < \mathbb{R}^n$ IS A SUBSPACE,
$\|\mathrm{PROJ}_V(\underline{x})\| \le \|\underline{x}\|$
PROOF: $\underline{x} = \underline{x}^{\parallel} + \underline{x}^{\perp}$, $\underline{x}^{\parallel} = \mathrm{PROJ}_V(\underline{x})$, $\underline{x}^{\perp} \perp \underline{x}^{\parallel}$.
THUS $\|\underline{x}\|^2 = \|\underline{x}^{\parallel}\|^2 + \|\underline{x}^{\perp}\|^2$
$\|\underline{x}\| \ge \|\underline{x}^{\parallel}\|$.
$\|\underline{x}\|$ = "LENGTH OF $\underline{x}$".
$\underline{x}$
$\underline{x}^{\perp}$
$\underline{x}^{\parallel}$
$\|\underline{x}\|^2 = \|\underline{x}^{\parallel}\|^2 + \|\underline{x}^{\perp}\|^2$.

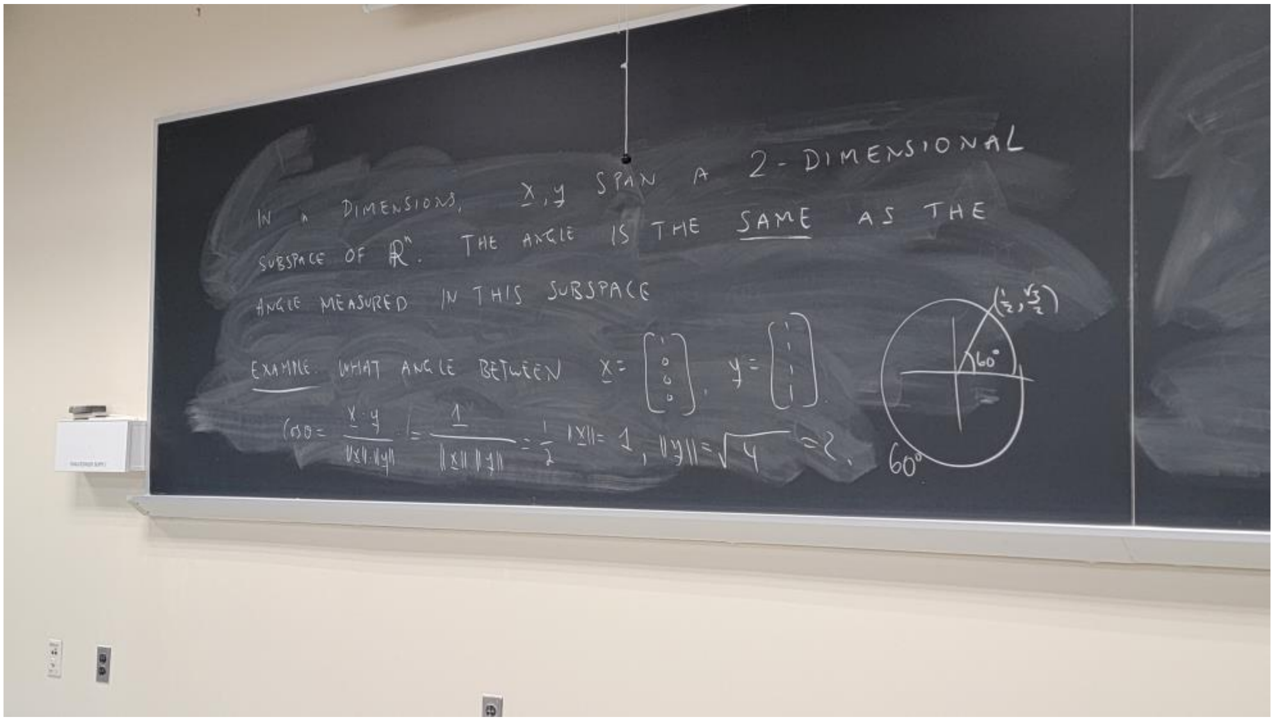

CAUCHY-SCHWARZ INEQUALITY.
IF $\underline{x},\underline{y} \in \mathbb{R}^n$, $|\underline{x}\cdot\underline{y}| \le \|\underline{x}\|\cdot\|\underline{y}\|$
EQUALITY IF AND ONLY IF THE VECTORS ARE PARALLEL.
PROOF: CONSIDER $f(t) = \|\underline{x} - t\underline{y}\|^2 = (\underline{x} - t\underline{y})\cdot(\underline{x} - t\underline{y})$
$= \underline{x}\cdot\underline{x} - 2t\,\underline{x}\cdot\underline{y} + t^2\,\underline{y}\cdot\underline{y}$.
$f'(t) = -2\,\underline{x}\cdot\underline{y} + 2t\|\underline{y}\|^2$

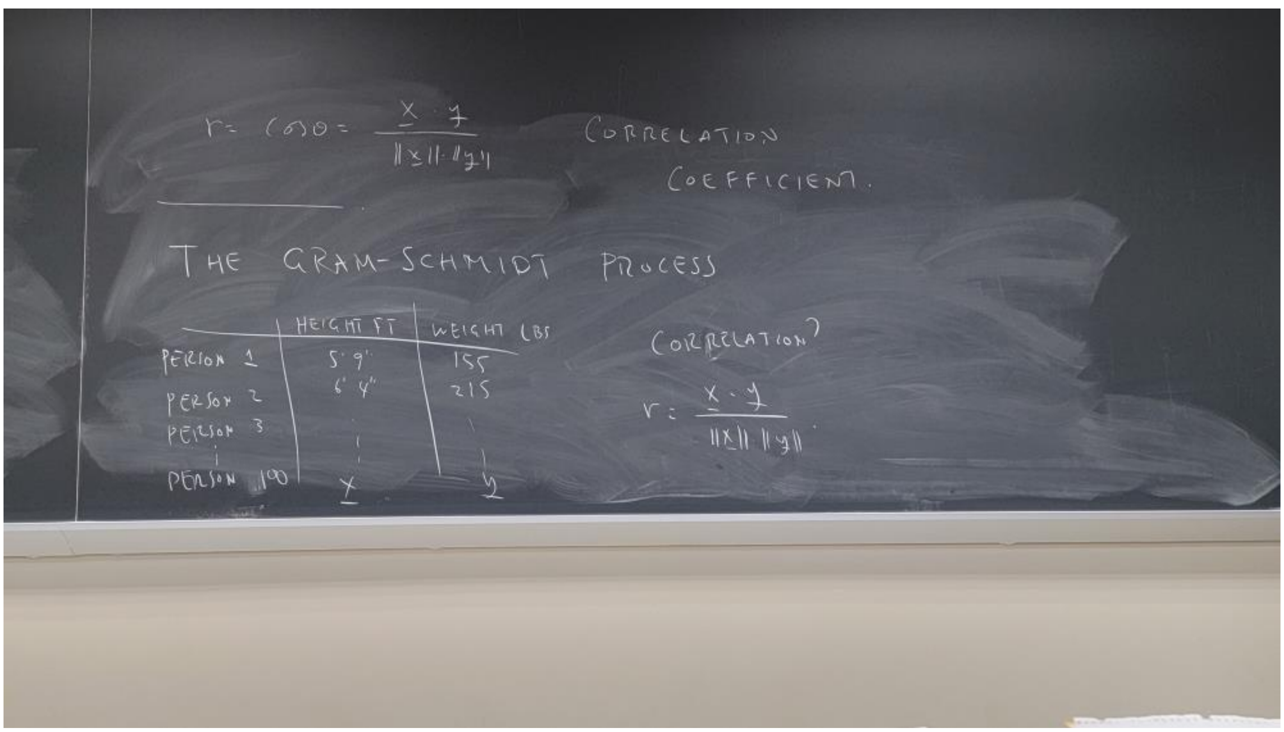

CAUCHY-SCHWARZ INEQUALITY.
IF x, y ∈ ℝⁿ, |x · y| ≤ ‖x‖ · ‖y‖
EQUALITY IF AND ONLY IF THE VECTORS ARE PARALLEL.
PROOF: CONSIDER f(t) = ‖x - ty‖² = (x - ty) · (x - ty)
= x · x - 2t x · y + t² y · y.
f′(t) = -2 x · y + 2t‖y‖².

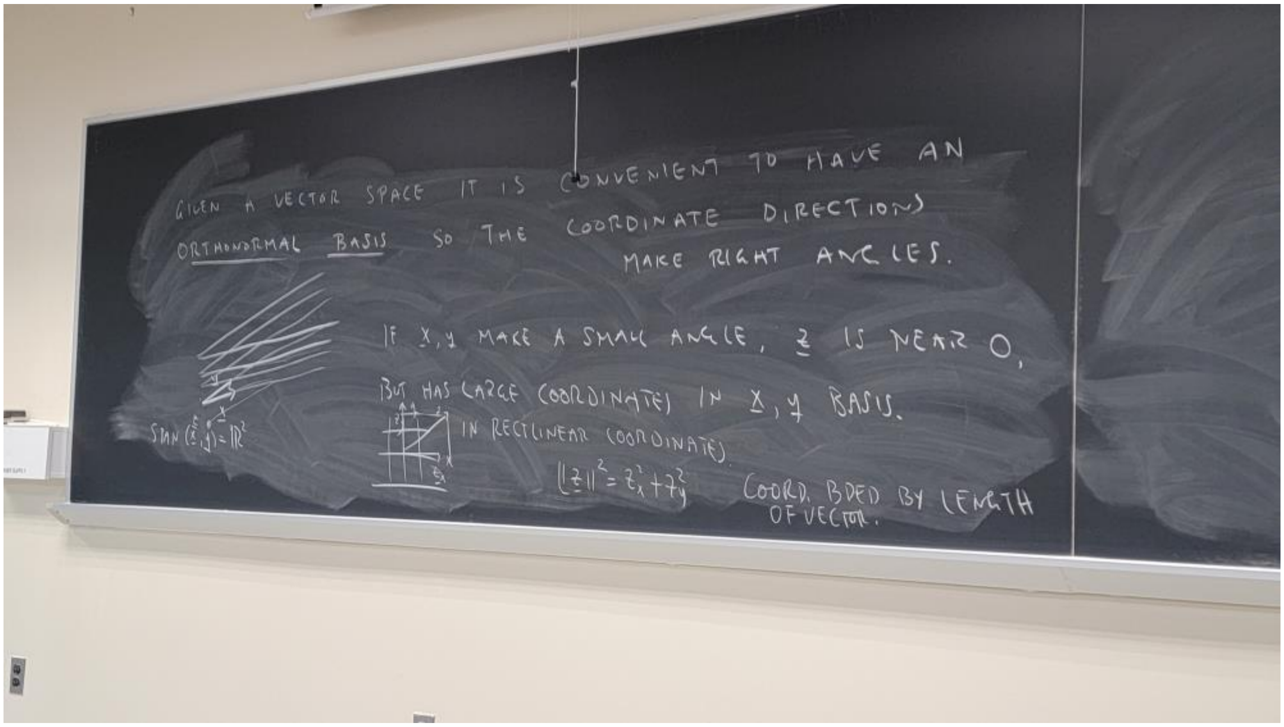

THE LENGTH IS MINIMIZED IF
x · y = t ‖y‖².
ASSUME y ≠ 0, SINCE CLAIM HOLDS IF y = 0.
t = x · y / ‖y‖².
0 ≤ ‖x‖² - 2t x · y + t²‖y‖² = ‖x‖² - |x · y|² / ‖y‖² ⇒ |x · y|² ≤ ‖x‖² ‖y‖².
EQUALITY IF x - ty = 0 ⇒ PARALLEL. □

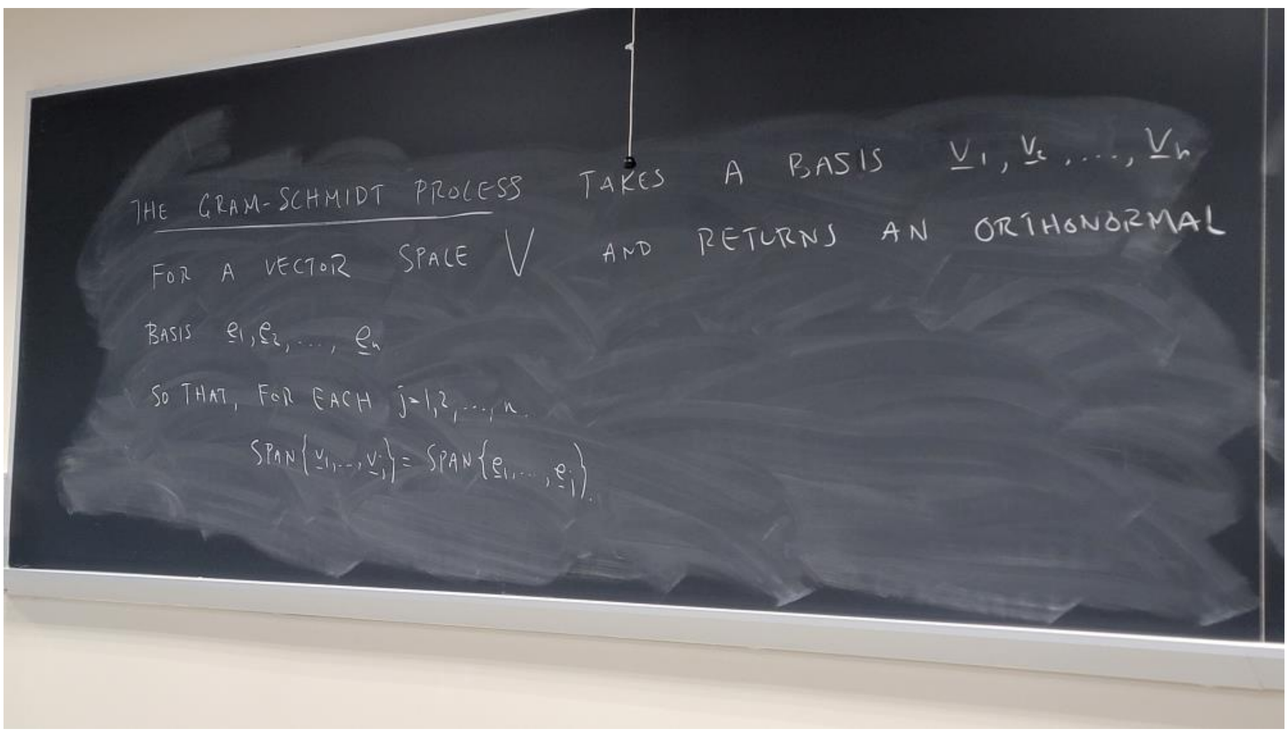

GEOMETRIC INTERPRETATION.
GIVEN VECTORS x, y, BOTH NON-ZERO, DEFINE THE
ANGLE BETWEEN THEM TO BE
cos θ = x · y / ‖x‖·‖y‖ , θ = ANGLE.
THIS DEFINES A NOTION OF ANGLE IN ℝⁿ.

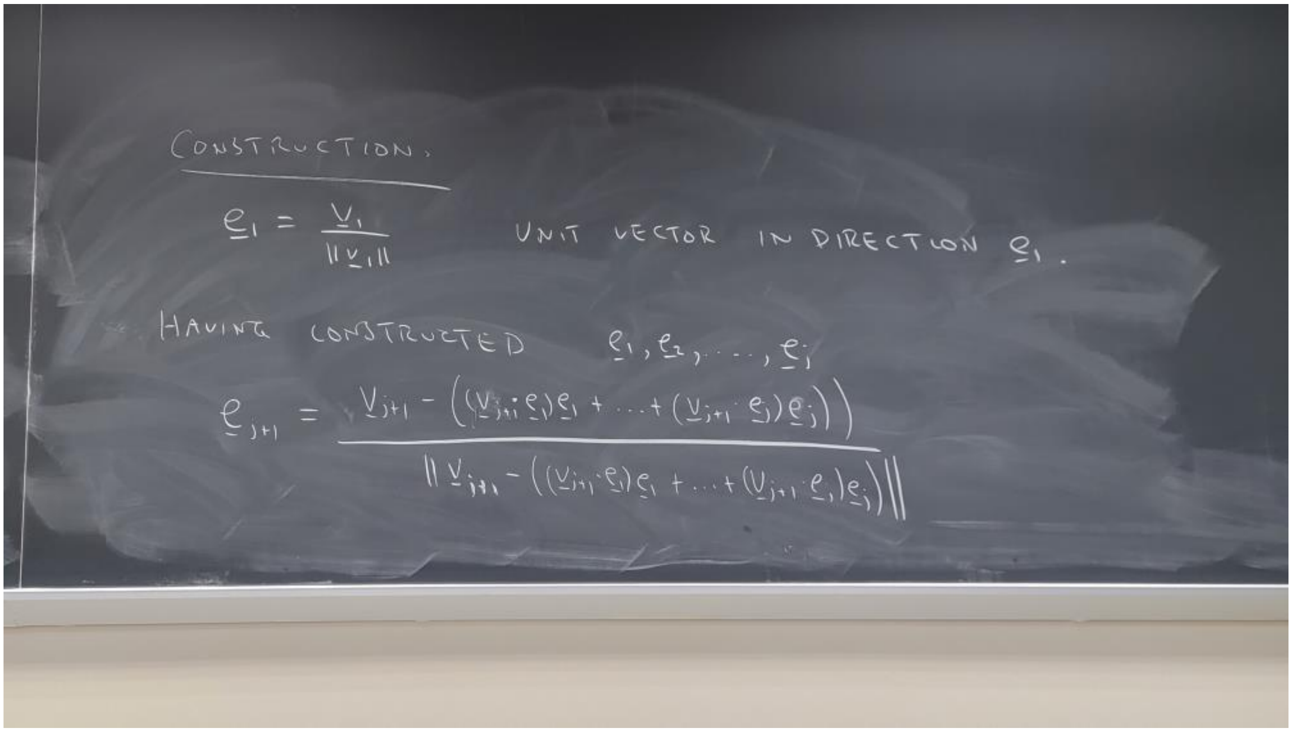

IN ℝ²
|cos θ| = | PROJ x̂ ŷ |.
= | x̂ · ŷ |.
LENGTH OF VECTORS DOESN'T MATTER
WHEN MEASURING ANGLE. x̂ = x / ‖x‖ , ŷ = y / ‖y‖
x̂, ŷ LENGTH 1.

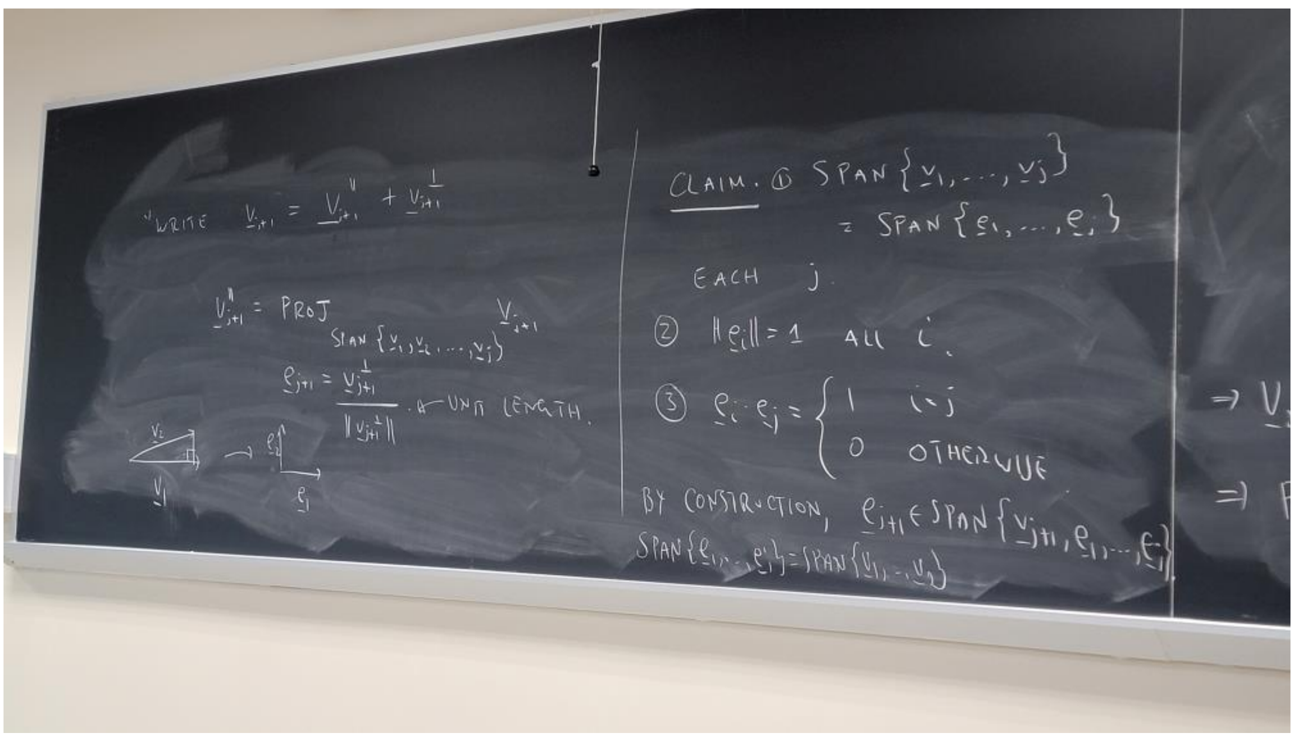

IN n DIMENSIONS, x, y SPAN A 2-DIMENSIONAL
SUBSPACE OF ℝⁿ. THE ANGLE IS THE SAME AS THE
ANGLE MEASURED IN THIS SUBSPACE
EXAMPLE WHAT ANGLE BETWEEN x = [1, 0, 0, 0], y = [1, 1, 1, 1]
cos θ = x·y / ‖x‖·‖y‖ = 1 / ‖x‖ ‖y‖ = 1/2 ‖x‖ = 1, ‖y‖ = √4 = 2.
60°
60°
(1/2, √3/2)

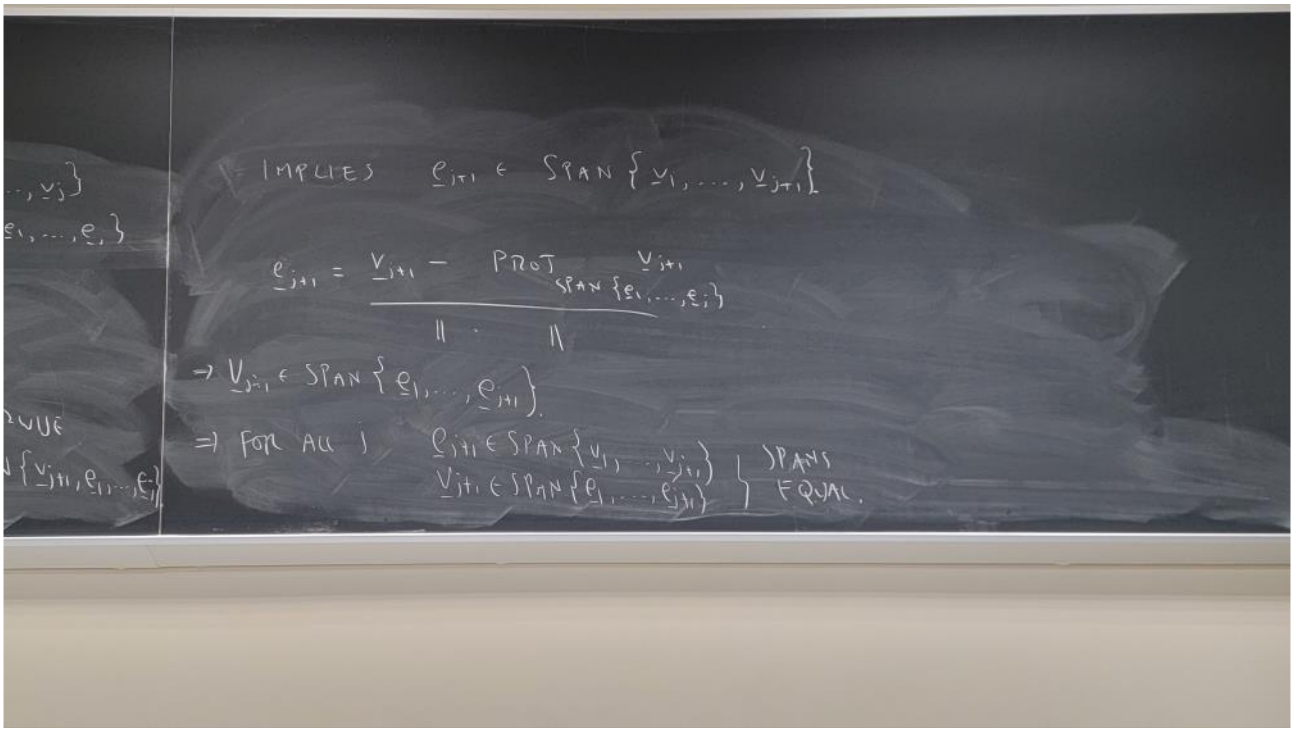

r = cos θ = x · y / ‖x‖·‖y‖
CORRELATION COEFFICIENT.
THE GRAM-SCHMIDT PROCESS
HEIGHT FT
WEIGHT LBS
PERSON 1
5' 9"
155
PERSON 2
6' 4"
215
PERSON 3
PERSON 100
x
y
CORRELATION?
r = x · y / ‖x‖ ‖y‖

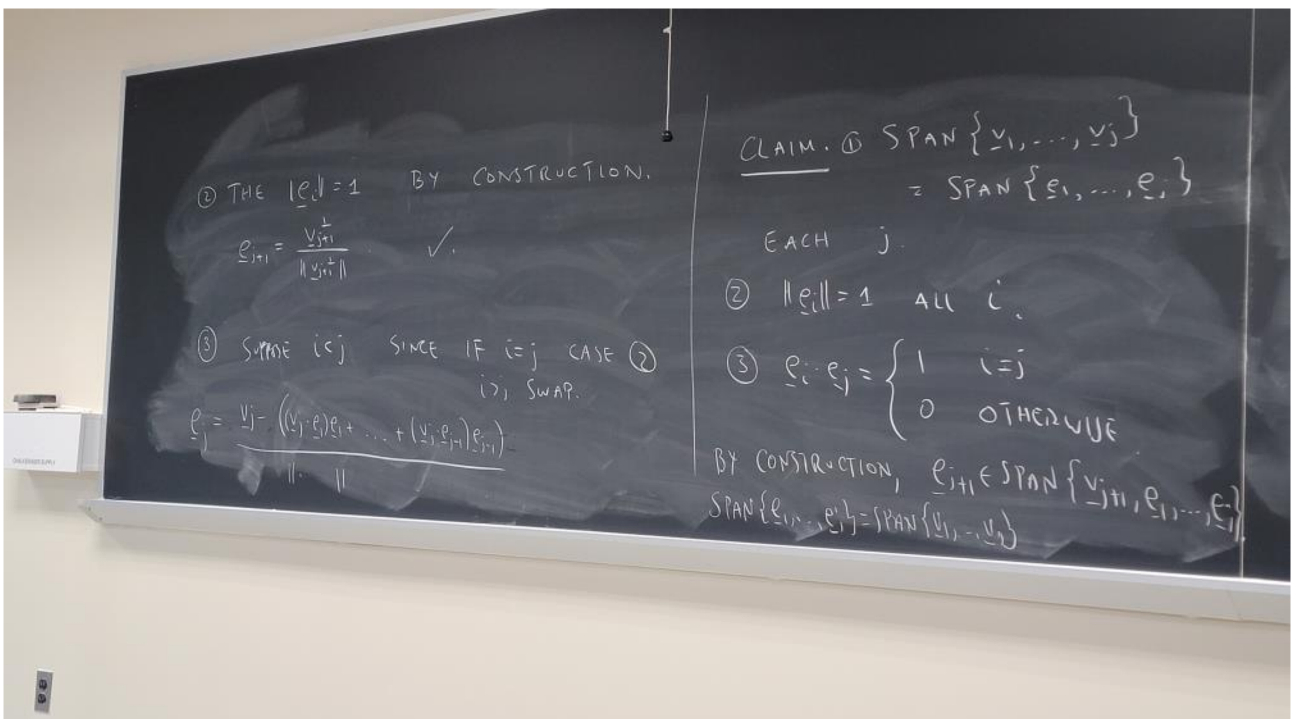

GIVEN A VECTOR SPACE IT IS CONVENIENT TO HAVE AN
ORTHONORMAL BASIS SO THE COORDINATE DIRECTIONS
MAKE RIGHT ANGLES.
IF X, Y MAKE A SMALL ANGLE, Z IS NEAR 0,
BUT HAS LARGE COORDINATES IN X, Y BASIS.
SPAN (X, Y) = ℝ²
IN RECTILINEAR COORDINATES
‖Z‖² = Zx² + Zy²
COORD. BDED BY LENGTH OF VECTOR.

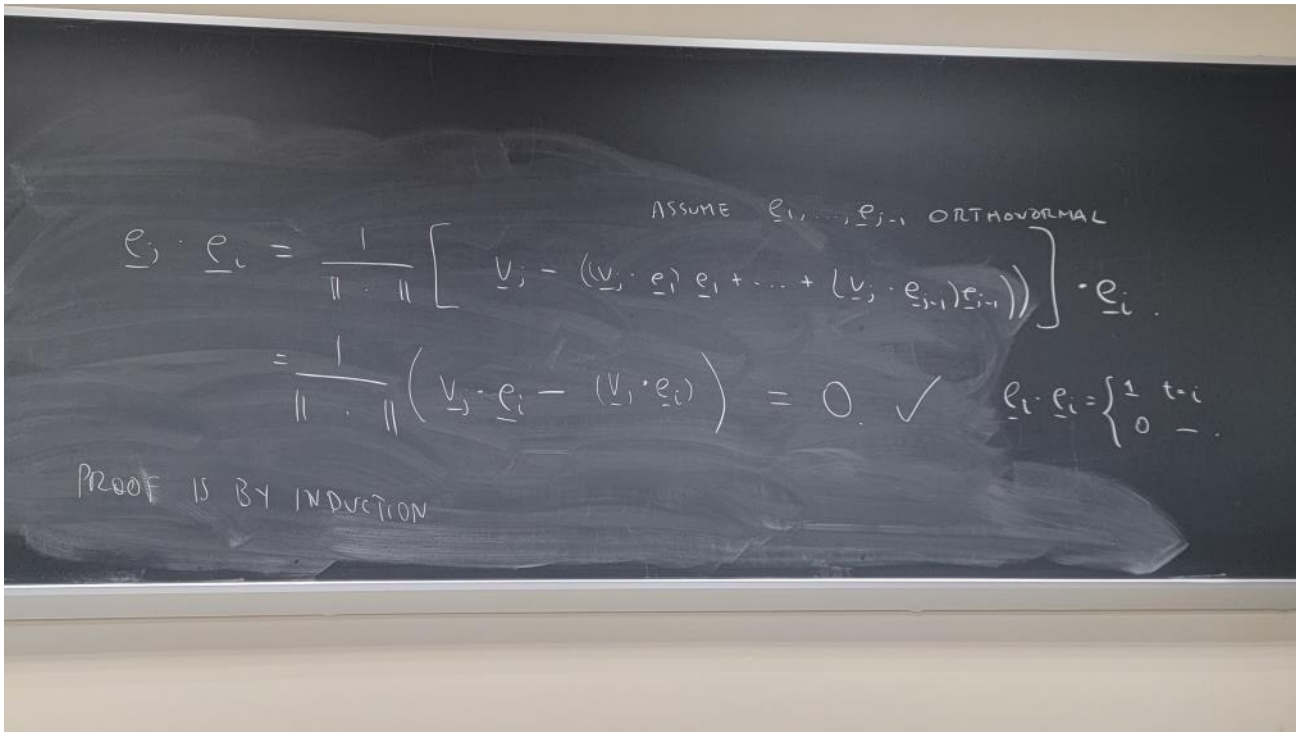

THE GRAM-SCHMIDT PROCESS TAKES A BASIS V1, V2, ..., Vn
FOR A VECTOR SPACE V AND RETURNS AN ORTHONORMAL
BASIS e1, e2, ..., en
SO THAT, FOR EACH j = 1, 2, ..., n.
SPAN{v1, ..., vj} = SPAN{e1, ..., ej}.

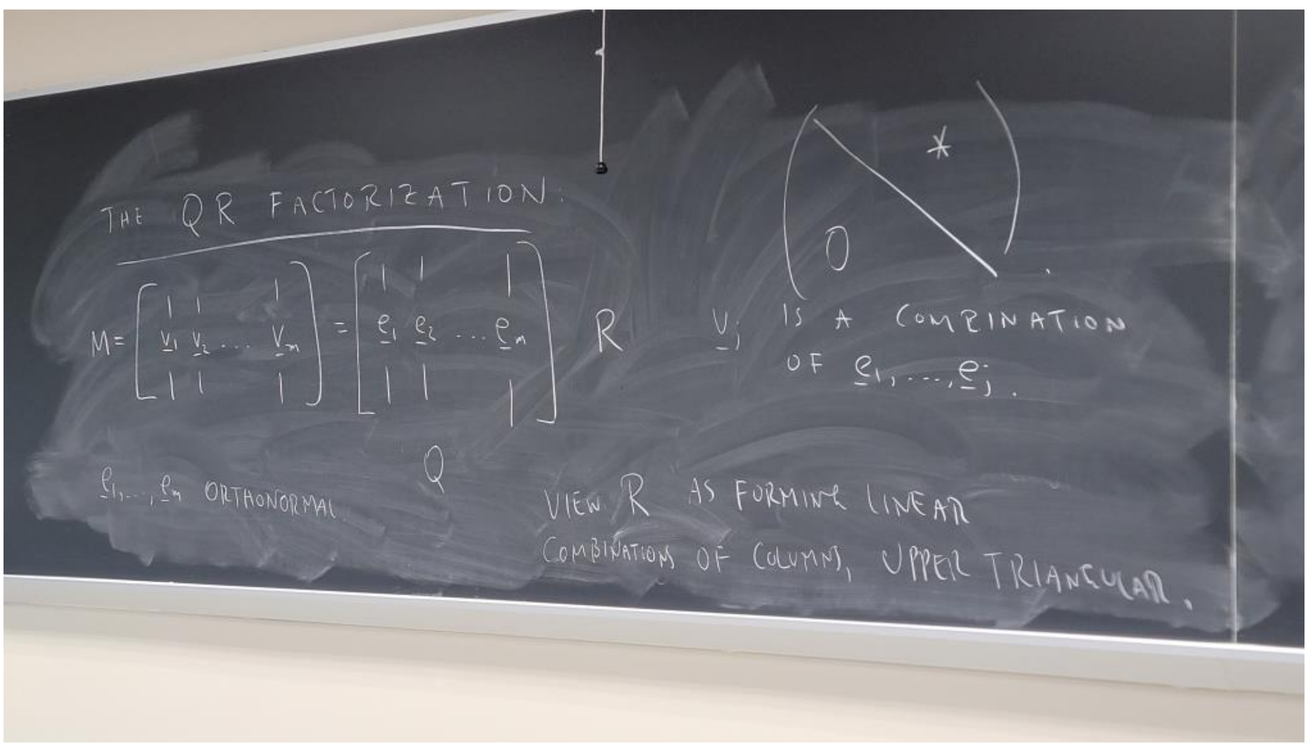

CONSTRUCTION.
$e_1 = \frac{v_1}{\|v_1\|}$ UNIT VECTOR IN DIRECTION e_1.
HAVING CONSTRUCTED $e_1, e_2, \ldots, e_j$
$e_{j+1} = \frac{v_{j+1} - ((v_{j+1} \cdot e_1)e_1 + \ldots + (v_{j+1} \cdot e_j)e_j)}{\| v_{j+1} - ((v_{j+1} \cdot e_1)e_1 + \ldots + (v_{j+1} \cdot e_j)e_j) \|}$

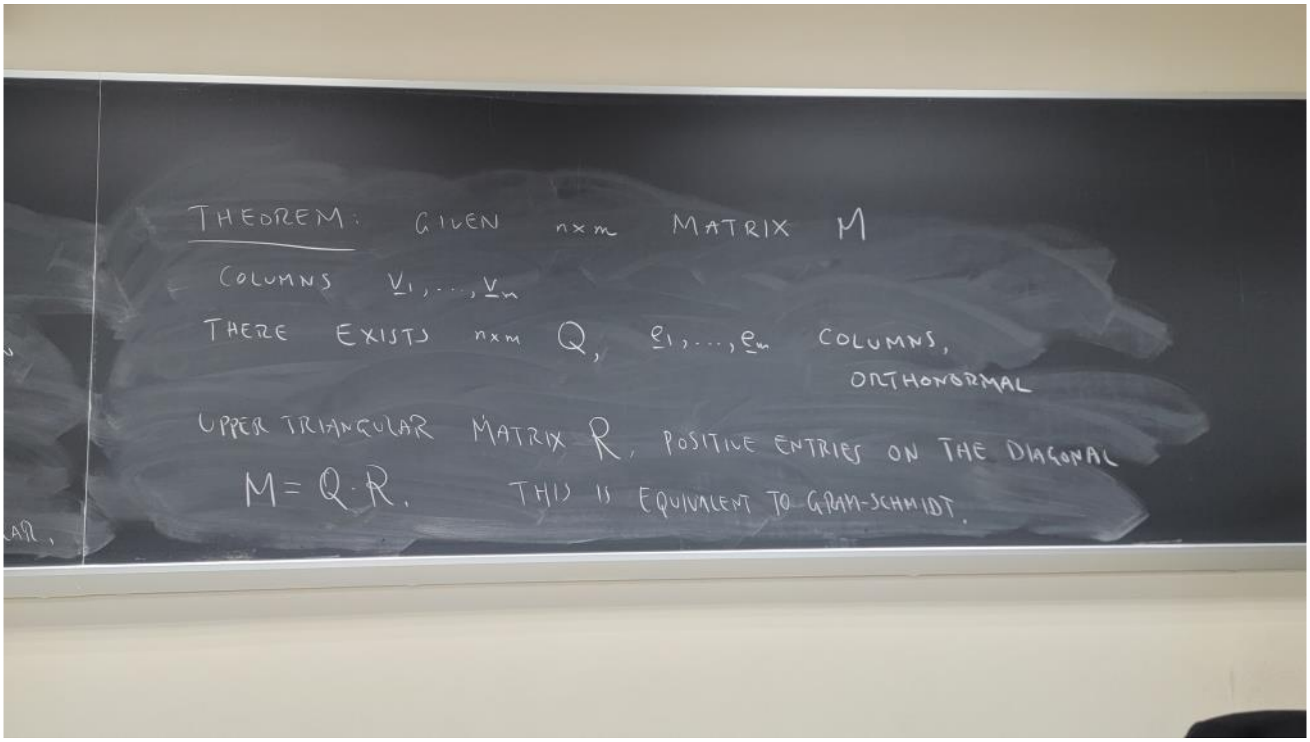

WRITE $v_{j+1} = v_{j+1}^{\parallel} + v_{j+1}^{\perp}$
$v_{j+1}^{\parallel} = \mathrm{PROJ}_{\mathrm{SPAN}\{v_1, v_2, \ldots, v_j\}} v_{j+1}$
$e_{j+1} = \frac{v_{j+1}^{\perp}}{\|v_{j+1}^{\perp}\|}$ ← UNIT LENGTH.
v_2
v_1
e_2
e_1
CLAIM. ① $\mathrm{SPAN}\{v_1, \ldots, v_j\} = \mathrm{SPAN}\{e_1, \ldots, e_j\}$
EACH j.
② $\|e_i\| = 1$ ALL i.
③ $e_i \cdot e_j = \begin{cases} 1 & i = j \\ 0 & \text{OTHERWISE} \end{cases}$
BY CONSTRUCTION, $e_{j+1} \in \mathrm{SPAN}\{v_{j+1}, e_1, \ldots, e_j\}$.
$\mathrm{SPAN}\{e_1, \ldots, e_j\} = \mathrm{SPAN}\{v_1, \ldots, v_j\}$

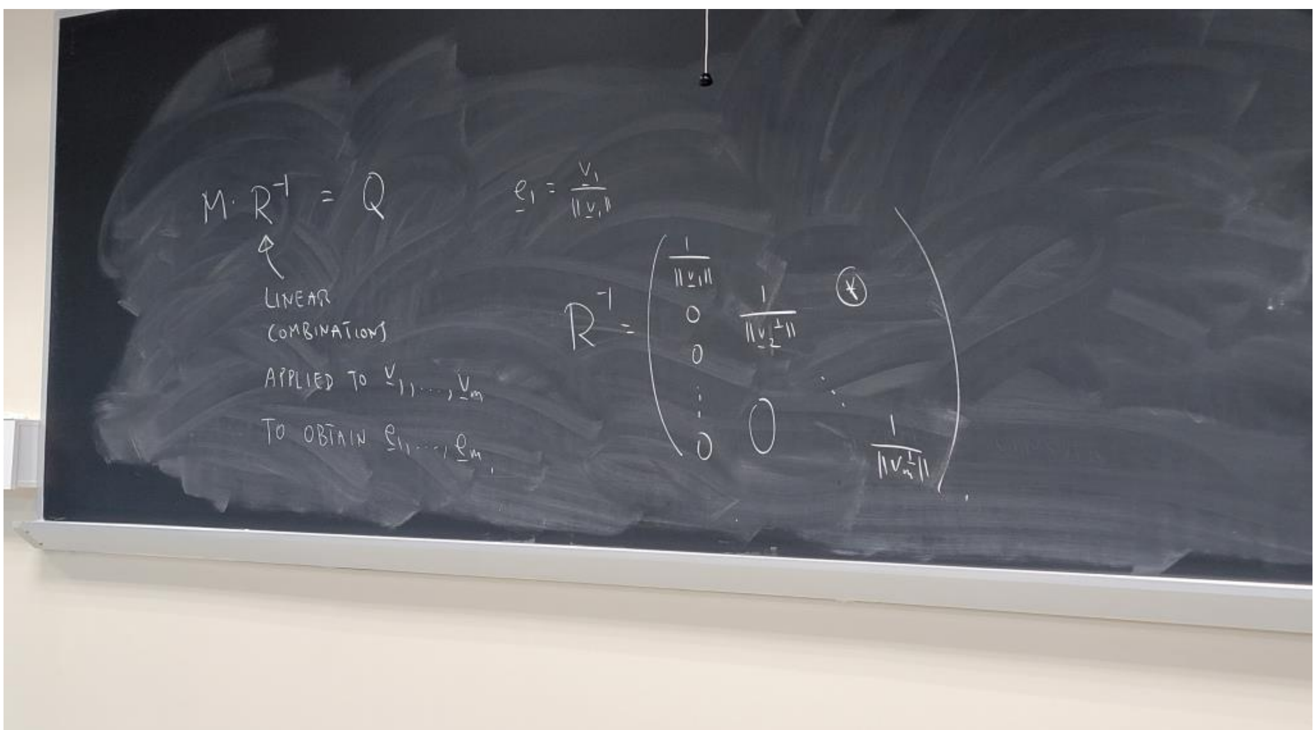

IMPLIES e_{j+1} ∈ SPAN {v_1, ..., v_{j+1}}
e_{j+1} = (v_{j+1} − PROJ_{SPAN {e_1, ..., e_j}} v_{j+1}) / ‖ · ‖
⇒ v_{j+1} ∈ SPAN {e_1, ..., e_{j+1}}
⇒ FOR ALL j e_{j+1} ∈ SPAN {v_1, ..., v_{j+1}}
v_{j+1} ∈ SPAN {e_1, ..., e_{j+1}}
SPANS EQUAL.

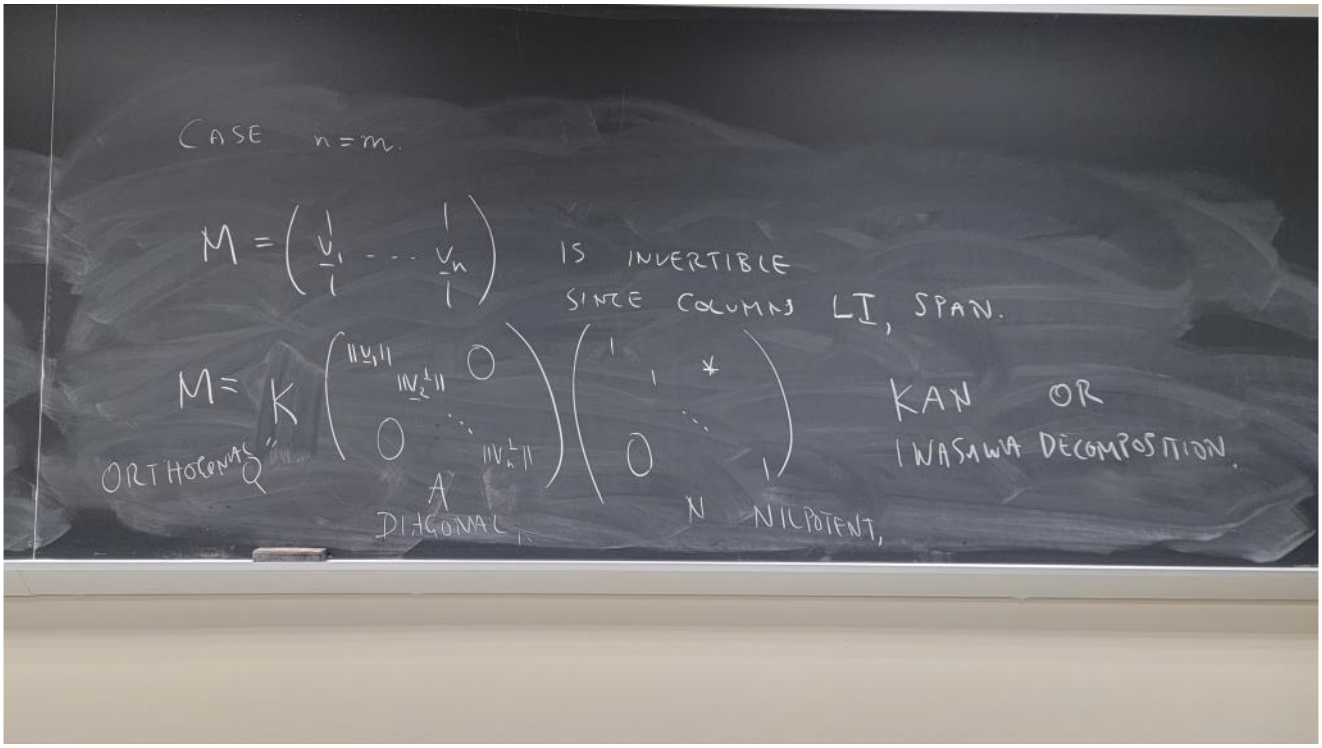

CLAIM. ① SPAN {v_1, ..., v_j}
= SPAN {e_1, ..., e_j}
EACH j.
② ‖e_i‖ = 1 ALL i.
③ e_i · e_j = { 1 i=j ; 0 OTHERWISE
BY CONSTRUCTION, e_{j+1} ∈ SPAN {v_{j+1}, e_1, ..., e_j}
SPAN {e_1, ..., e_j} = SPAN {v_1, ..., v_j}
② THE ‖e_i‖ = 1 BY CONSTRUCTION.
e_{j+1} = v_{j+1}^⊥ / ‖v_{j+1}^⊥‖ ✓.
③ SUPPOSE i<j SINCE IF i=j CASE ②
i>j SWAP.
e_j = (v_j − ((v_j · e_1)e_1 + ... + (v_j · e_{j−1})e_{j−1})) / ‖ · ‖

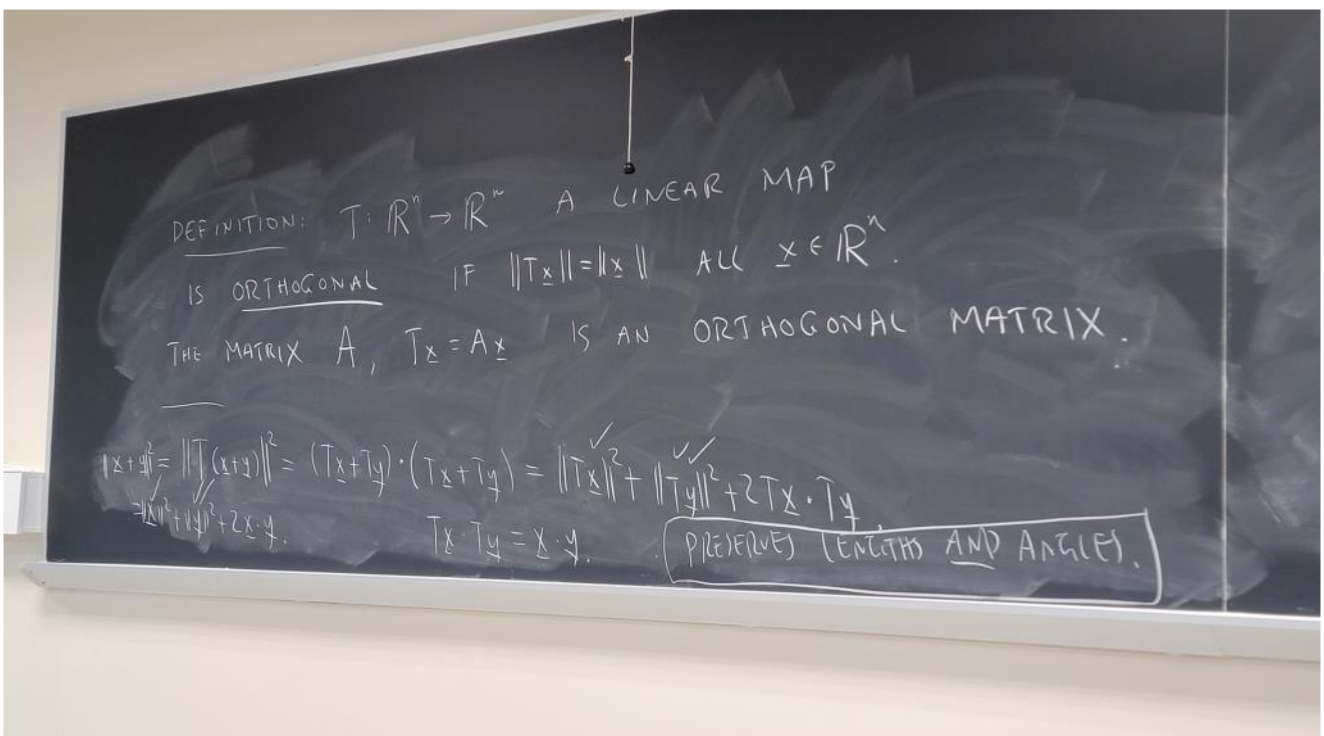

ASSUME e1, ..., ej-1 ORTHONORMAL
ej · ei = 1/|| · || [vj − ((vj · e1) e1 + ... + (vj · ej-1) ej-1)] · ei .
= 1/|| · || (vj · ei − (vj · ei)) = 0 ✓
PROOF IS BY INDUCTION

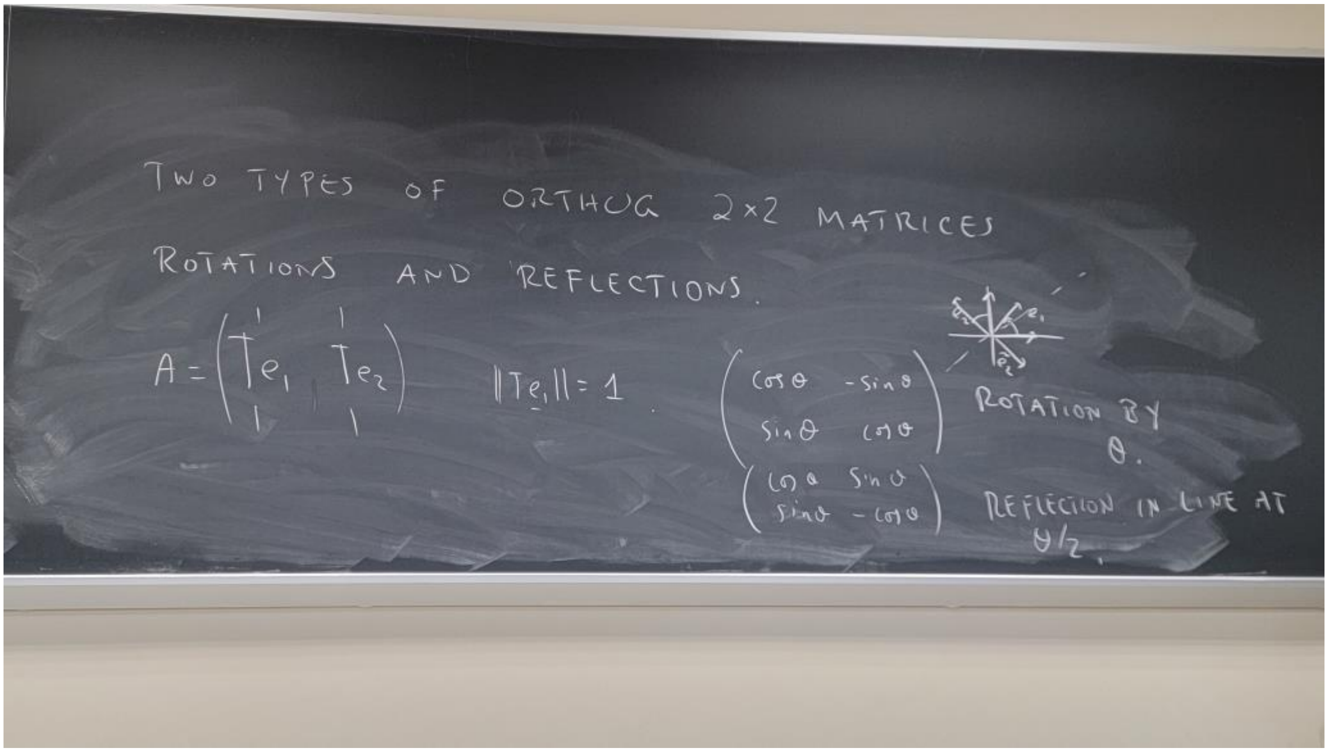

THE QR FACTORIZATION:
M = v1 v2 ... vn = e1 e2 ... en R
Q
e1, ..., en ORTHONORMAL
*
0
vj IS A COMBINATION OF e1, ..., ej .
VIEW R AS FORMING LINEAR
COMBINATIONS OF COLUMNS, UPPER TRIANGULAR.

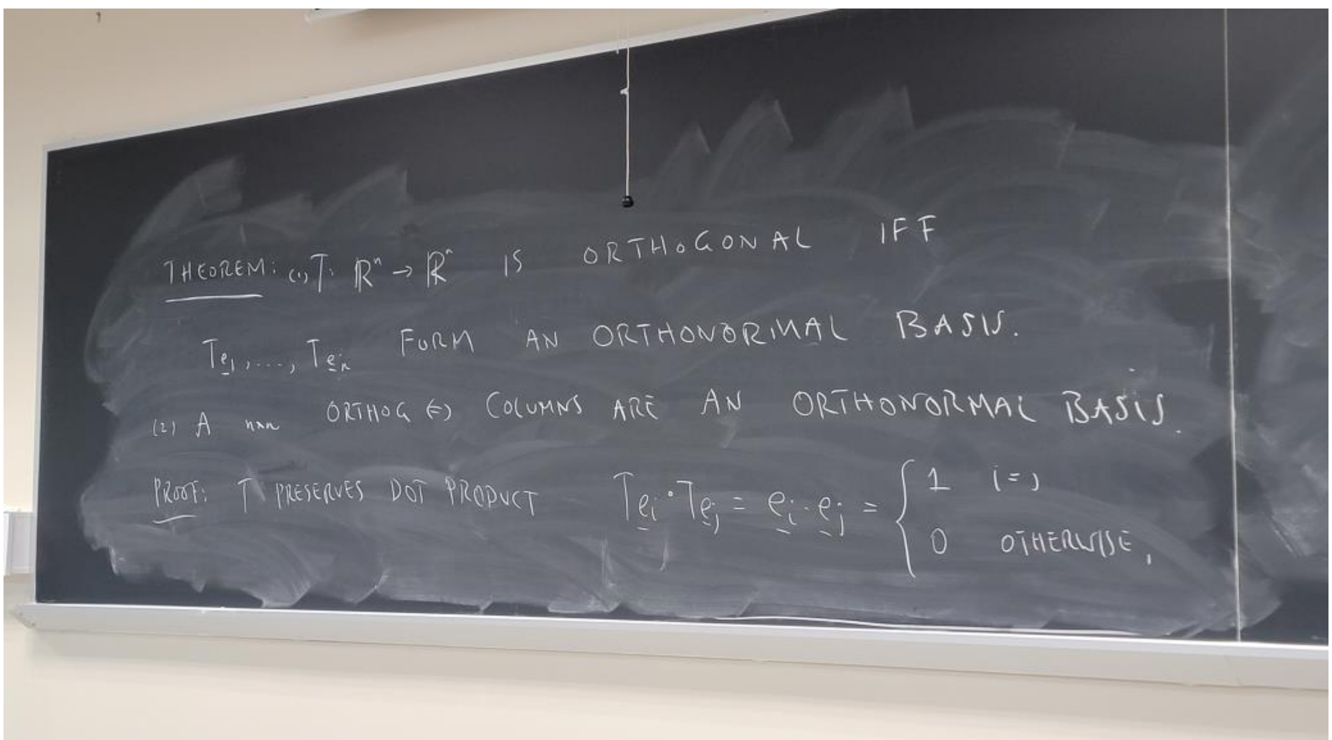

THEOREM: GIVEN n×m MATRIX M
COLUMNS $\underline{v}_1, \dots, \underline{v}_m$
THERE EXISTS n×m Q, $\underline{e}_1, \dots, \underline{e}_m$ COLUMNS, ORTHONORMAL
UPPER TRIANGULAR MATRIX R, POSITIVE ENTRIES ON THE DIAGONAL
M = Q·R. THIS IS EQUIVALENT TO GRAM-SCHMIDT.

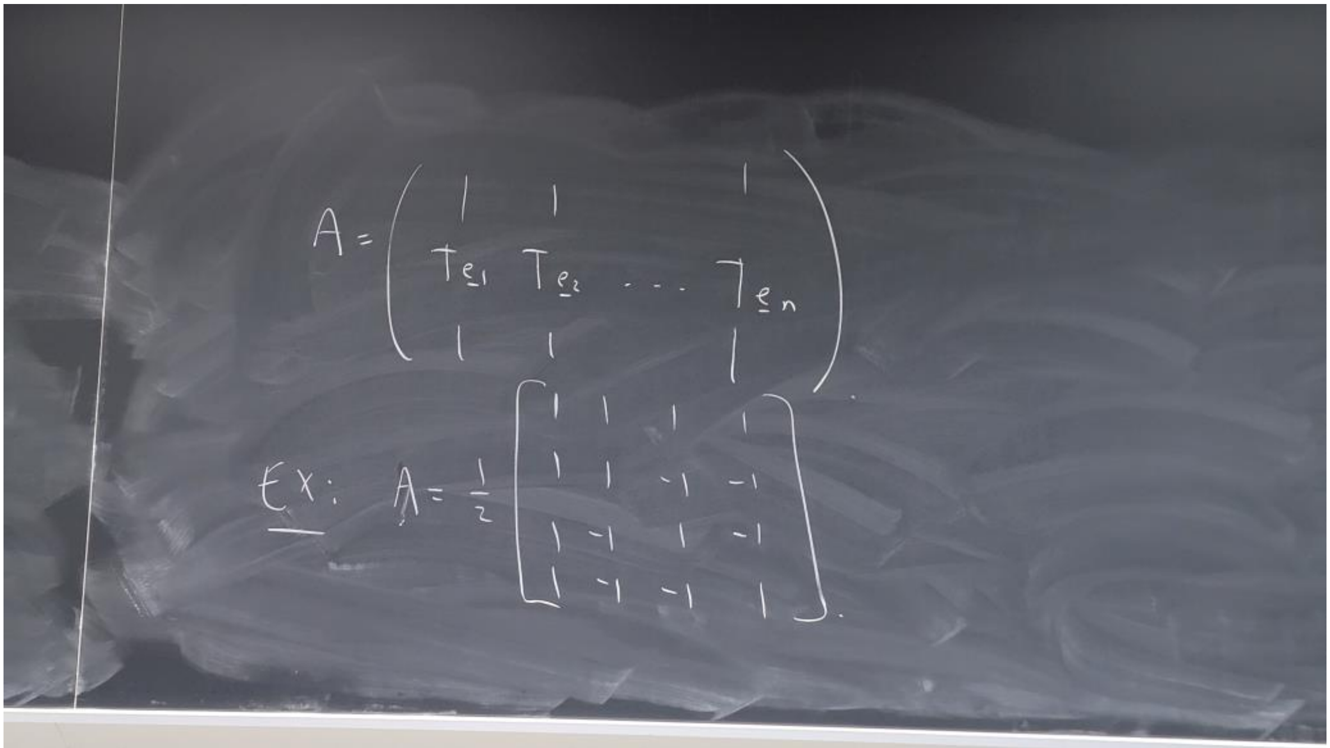

$M \cdot R^{-1} = Q$
$\underline{e}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|}$
LINEAR COMBINATIONS APPLIED TO $\underline{v}_1, \dots, \underline{v}_m$ TO OBTAIN $\underline{e}_1, \dots, \underline{e}_m$.
$R^{-1} = \begin{pmatrix} \frac{1}{\|\underline{v}_1\|} & & * \\ 0 & \frac{1}{\|\underline{v}_2^\perp\|} & \\ 0 & & \ddots \\ \vdots & 0 & \\ 0 & & \frac{1}{\|\underline{v}_m^\perp\|} \end{pmatrix}$.

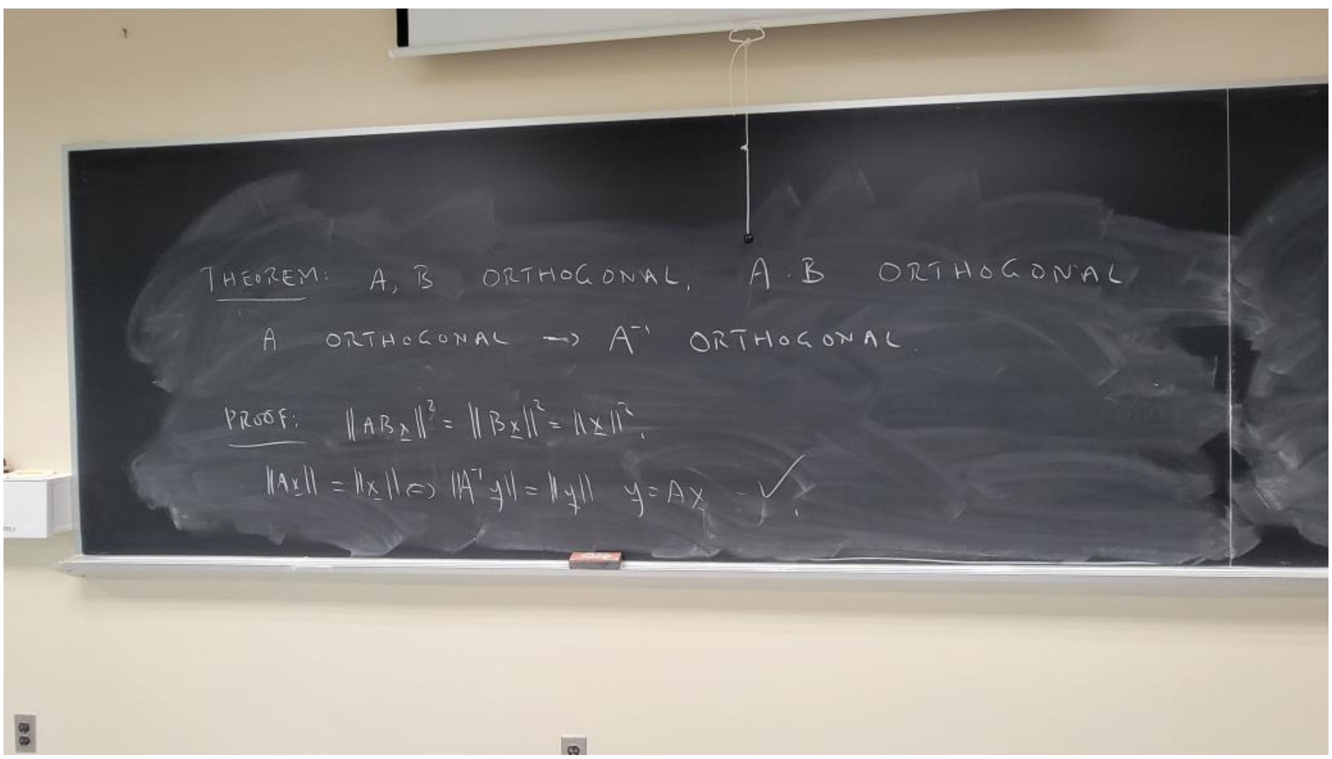

CASE n=m.
M = (v1 ... vn) IS INVERTIBLE
SINCE COLUMNS LI, SPAN.
M = K (||v1|| ... ||vn⊥||) (1 * ... 0 1)
KAN OR
IWASAWA DECOMPOSITION.
ORTHOGONAL Q
A
DIAGONAL,
N NILPOTENT,


DEFINITION: T: ℝⁿ → ℝⁿ A LINEAR MAP
IS ORTHOGONAL IF ||Tx|| = ||x|| ALL x ∈ ℝⁿ.
THE MATRIX A, Tx = Ax IS AN ORTHOGONAL MATRIX.
||x+y||² = ||T(x+y)||² = (Tx+Ty)·(Tx+Ty) = ||Tx||² + ||Ty||² + 2Tx·Ty
= ||x||² + ||y||² + 2x·y.
Tx·Ty = x·y.
PRESERVES LENGTHS AND ANGLES.


TWO TYPES OF ORTHOG 2×2 MATRICES
ROTATIONS AND REFLECTIONS.
A = (Te1 Te2)
‖Te1‖ = 1
(cos θ −sin θ / sin θ cos θ) ROTATION BY θ.
(cos θ sin θ / sin θ −cos θ) REFLECTION IN LINE AT θ/2.


THEOREM: (1) T: ℝⁿ → ℝⁿ IS ORTHOGONAL IFF
Te1, ..., Ten FORM AN ORTHONORMAL BASIS.
(2) A n×n ORTHOG ⇔ COLUMNS ARE AN ORTHONORMAL BASIS.
PROOF: T PRESERVES DOT PRODUCT Tei · Tej = ei · ej = { 1 i=j ; 0 OTHERWISE.


$A = \begin{pmatrix} | & | & & | \\ Te_1 & Te_2 & \cdots & Te_n \\ | & | & & | \end{pmatrix}.$
Ex: $A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$


THEOREM: A, B ORTHOGONAL, A · B ORTHOGONAL
A ORTHOGONAL → A^{-1} ORTHOGONAL.
PROOF: $\|AB\underline{x}\|^2 = \|B\underline{x}\|^2 = \|\underline{x}\|^2.$
$\|A\underline{x}\| = \|\underline{x}\| \Leftrightarrow \|A^{-1}\underline{y}\| = \|\underline{y}\| \quad \underline{y} = A\underline{x}$ ✓.